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# EXAMPLES

IN THE

# INTEGRAL CALCULUS,

COMPILED FOR

THE USE OF THE CADETS

AT THE

UNITED STATES NAVAL ACADEMY.

◆

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# THE INTEGRAL CALCULUS.

## DEFINITIONS.

**1.** In the differential calculus, we have to find the rates, or differentials, of given functions.

In the integral calculus, having given any differential, we have to determine the function of which this is the rate; this function is called the *integral* of the given differential.

The *integral sign* is $\int$; it is always written before an expression for a rate, the rate being generally expressed by a single variable and its differential; as,

$$\int x^2\,dx$$

which means that function of $x$ whose rate is $x^2\,dx$.

**2.** A *definite integral* is written with a number or letter, denoting a special value of the independent variable, at the bottom, and another at the top of the integral sign (these letters or numbers being called limits); thus,

$$\int_1^3 x^2\,dx$$

and indicates the amount by which a quantity, varying with the rate under the integral sign, actually varies while the independent variable passes from the lower to the upper limit. Thus, the above expression denotes the increment received by any quantity which has the rate $x^2\,dx$ while $x$ increases from 1 to 3.

**3.** An *indefinite integral* is simply a variable which varies with the rate expressed under the integral sign, and is written without limits; thus,

$$\int x^2\,dx$$

It will be seen that the definite integral may, then, be defined as the increment received by the indefinite integral while $x$ passes from the lower to the upper limit. The indefinite integral is thus a function of $x$, while the definite integral, being the increment received while $x$ passes from one limit to the other, is a function of these limits or special values of $x$. The indefinite integral, being then such a function as will have the given rate, will be known if we can recognize the differential as one derivable from any particular function.

Thus, $d\,(\frac{1}{3}x^3) = x^2\,dx$; hence $\frac{1}{3}\,x^3$ is an integral of $x^2\,dx$, *i. e.*, a quantity which has that rate; and

$$\int_1^3 x^2\,dx = 8\tfrac{2}{3}$$

because $\frac{1}{3}\,x^3$ increases from $\frac{1}{3}$ to 9 as $x$ increases from 1 to 3. The indefinite integral is indefinite, not only because it is a variable or function of $x$, but also for the following reason: $\frac{1}{3}\,x^3$ is not the only quantity which has the rate $x^2\,dx$; for if $c$ denote a constant, $\frac{1}{3}\,x^3 + c$ has the same rate; hence the indefinite integral

$$\int x^2\,dx = \tfrac{1}{3}\,x^3 + c$$

which includes all functions of $x$ which have the rate $x^2\,dx$. $c$ is wholly indefinite in value, and is called the ***constant of integration.***

**4.** The definite integral is derived from the indefinite by computing the value for the upper limit and subtracting from it that for the lower limit; this, evidently, is the increment received by the indefinite integral as $x$ passes from the lower to the upper limit,

$$\int_1^3 x^2\,dx = \tfrac{1}{3}\,.\,3^3 - \tfrac{1}{3}\,.\,1^3 = 8\tfrac{2}{3}$$

It will be seen that, if we had used the general form

$$\int x^2\,dx = \tfrac{1}{3}\,x^3 + c$$

we should have obtained the same value, $c$ disappearing by subtraction.

**5.** A ***particular integral*** is a particular value of the indefinite integral obtained by giving to the constant $c$ some special value. The particular value of the constant is usually determined by the condition that the integral shall have a certain value for a given value of the independent variable. Thus, the space through which a body falls freely *from rest* in $t$ seconds is a particular integral of $gtdt$, this being the expression for the rate at any time $t$ (from rest). It is, in fact, that particular integral which is zero when $t = 0$, since the position of rest is our origin of distances. This particular integral is denoted by

$$\int_0 gtdt$$

therefore we write

$$s = \int_0 gtdt$$

to express that $s$ is measured from the position of rest. An integral with a lower limit expressed, but without an upper limit, is then a function of the independent variable, and is the same as the indefinite integral minus its value at the lower limit, the constant of integration being thus eliminated. Thus,

$$\int t\,dt = \tfrac{1}{2}t^2 + c$$

$$\int_0 t\,dt = \tfrac{1}{2}t^2$$

hence, in above example,

$$s = \tfrac{1}{2}gt^2$$

**6.** From the notation employed for definite integrals, we easily see that

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

and

$$\int_a^a f(x)\,dx = 0$$

## ELEMENTARY PROPOSITIONS RELATIVE TO INTEGRALS.

**7.** The three elementary propositions of differential calculus; viz,

I, $d\,(x + h) = dx$
II, $d\,(mx) = m\,dx$
III, $d\,(x + y + \&c.) = dx + dy + \&c.$

give three propositions respecting integrals.

I shows that

$$\int dx = \int d\,(x + h) = x + h$$

or

$$\int dx = x + c$$

that is, points out the necessity of adding a constant of integration to get the general value of the indefinite integral.

II gives

$$\int_a^b m\,dx = mb - ma = m\,(b - a) = m\int_a^b dx$$

from which, as $x$ may represent any variable, we find that a constant factor under the integral sign may be removed and written as a coefficient of the integral; thus,

$$\int m\,dx = m\int dx$$

We may also, when desirable, introduce a constant factor under the integral sign and its reciprocal outside of the integral sign, for

$$\int dx = \int \frac{1}{m}\,m\,dx = \frac{1}{m}\int m\,dx$$

thus

$$\int x^4\,dx = \tfrac{1}{5}\int 5x^4\,dx = \tfrac{1}{5}\,x^5 + c$$

since we know that

$$d\,(x)^5 = 5x^4\,dx$$

III gives the following principle of integration, since

$$d\,(x + y + \&\text{c.}) = dx + dy + \&\text{c.}$$

$$\int (dx + dy + \&\text{c.}) = \int dx + \int dy + \&\text{c.} + c$$

A constant of integration may be considered as arising from each term of the second member, all of which constants unite into a single one. This expresses that a polynomial differential expression is integrated by treating each term separately. In the above expression of the principle, it is impossible to use limits, as the same limits cannot be values of each of the variables; but if, which is generally the case, we have a series of terms, each a function of the same variable, limits may be used on both sides; thus,

$$\int_a^b (\log x + m\cos x + 5)\,dx =$$

$$\int_a^b \log x\,dx + m\int_a^b \cos x\,dx + 5\,(b - a)$$

## FUNDAMENTAL INTEGRALS.

**8.** The fundamental differentials give the following integrals, on which all other integrable forms depend. The constant of integration, though omitted hereafter, must be considered as implied in all indefinite integrals.

1. $\int x^m \, dx = \frac{x^{m+1}}{m+1}$

2. $\int \frac{dx}{x} = \log x$

3. $\int a^x \, dx = \frac{a^x}{\log a}$

4. $\int \cos x \, dx = \sin x$

5. $\int \sin x \, dx = -\cos x$

6. $\int \sec^2 x \, dx = \tan x$

7. $\int \operatorname{cosec}^2 x \, dx = -\cot x$

8. $\int \tan x \sec x \, dx = \sec x$

9. $\int \cot x \operatorname{cosec} x \, dx = -\operatorname{cosec} x$

10. $\int \frac{dx}{\sqrt{(a^2 - m^2x^2)}} = \frac{1}{m} \sin^{-1}\left[\frac{mx}{a}\right]$

11. $\int \frac{dx}{a^2 + m^2x^2} = \frac{1}{am} \tan^{-1}\left[\frac{mx}{a}\right]$

12. $\int \frac{dx}{x\sqrt{(m^2x^2 - a^2)}} = \frac{1}{a} \sec^{-1}\left[\frac{mx}{a}\right]$

13. $\int \frac{dx}{\sqrt{(2ax - mx^2)}} = \frac{1}{\sqrt{m}} \operatorname{vers}^{-1}\left[\frac{mx}{a}\right]$

Equation (1) is to be used when the value of $m$ is fractional or negative, except in the case where $m = -1$, when it fails to give an intelligible result, and equation (2) must be used.

## DIRECT INTEGRATION.

**9.** An expression is said to be *directly integrable* when it obviously corresponds in form with one of the above fundamental integrals, some other variable and its differential merely taking the place of $x$ and $dx$*;* for example,

$$\int \frac{\cos x \, dx}{\sin x} = \int \frac{d(\sin x)}{\sin x} = \log(\sin x)$$

It is also said to be directly integrable when it can be made to correspond to one of the fundamental forms by merely introducing a constant factor under the integral sign; thus,

$$\int \sqrt{(a - bx^2)}\, x\, dx = -\frac{1}{2b}\int (a - bx^2)^{\frac{1}{2}}(-2bx\, dx)$$
$$= -\frac{1}{3b}(a - bx^2)^{\frac{3}{2}}$$

## EXAMPLES.

### I.

1. $\int x \sqrt{(a^2 + x^2)}\, dx = \frac{1}{3}(a^2 + x^2)^{\frac{3}{2}}$
2. $\int \frac{x\, dx}{a^2 + x^2} = \log (a^2 + x^2)^{\frac{1}{2}}$
3. $\int \frac{x^2\, dx}{a - x^3} = \log \frac{1}{(a - x^3)^{\frac{1}{3}}}$
4. $\int (a^3 - x^3)^{\frac{4}{3}}\, x^2\, dx = -\frac{1}{7}(a^3 - x^3)^{\frac{7}{3}}$
5. $\int x \sqrt{(a^2 - b^2x^2)}\, dx = -\frac{1}{3b^2}(a^2 - b^2x^2)^{\frac{3}{2}}$
6. $\int \frac{dx}{x^2} = -\frac{1}{x}$
7. $\int \frac{dx}{(a - x)^2} = \frac{1}{a - x}$
8. $\int \frac{x\, dx}{\sqrt{(a^2 + x^2)}} = \sqrt{(a^2 + x^2)}$
9. $\int \frac{x^2\, dx}{\sqrt{(a^2 - bx^3)}} = -\frac{2}{3b}\sqrt{(a^2 - bx^3)}$
10. $\int (a + bx)^2\, dx = \frac{1}{3b}(a + bx)^3$
11. $\int (a + 3x^2)^3\, x\, dx = \frac{1}{24}(a + 3x^2)^4$
12. $\int \frac{dx}{x^5} = -\frac{1}{4x^4}$
13. $\int (a + bx^{n+1})^{n-1}\, x^n\, dx = \frac{(a + bx^{n+1})^n}{n\,(n+1)\,b}$
14. $\int \frac{2ax - x^2}{(3ax^2 - x^3)^{\frac{1}{3}}}\, dx = \frac{1}{2}(3ax^2 - x^3)^{\frac{2}{3}}$
15. $\int \left(\frac{12}{x^2} - \frac{3}{x^4}\right) dx = \frac{1}{x^3} - \frac{12}{x}$

16. $\int \frac{2\sqrt{(x)}\,dx}{1-x^{\frac{3}{2}}} = \log \frac{1}{(1-x^{\frac{3}{2}})^{\frac{4}{3}}}$

17. $\int \frac{7x\,dx}{8a-3x^2} = \log \frac{1}{(8a-3x^2)^{\frac{7}{6}}}$

18. $\int (2x^{\frac{2}{3}} + x^{-1})\,dx = \frac{6x^{\frac{5}{3}}}{5} + \log x$

19. $\int \frac{x^2-4x+3}{x^3-6x^2+9x}\,dx = \log\,(x^3-6x^2+9x)^{\frac{1}{3}}$

20. $\int \left(a - \frac{b}{x^3} + cx^{\frac{3}{2}}\right) dx = ax + \frac{2cx^{\frac{5}{2}}}{5} + \frac{b}{2x^2}$

21. $\int \frac{x^{n-1}\,dx}{(a+bx^n)^m} = -\frac{1}{n\,b\,(m-1)\,(a+bx^n)^{m-1}}$

22. $\int \frac{(a-x)\,dx}{\sqrt{(2ax-x^2)}} = \sqrt{(2ax-x^2)}$

23. $\int 3\,(a^4-x^4)^{\frac{5}{3}}\,x^3\,dx = -\frac{9}{32}\,(a^4-x^4)^{\frac{8}{3}}$

24. $\int \frac{a^2+2ax}{\sqrt{(ax+x^2)}}\,dx = 2a\sqrt{(ax+x^2)}$

25. $\int \sin\,(2x)\,dx = -\,\frac{1}{2}\cos\,(2x)$

26. $\int \sin^3 x \cos x\,dx = \frac{\sin^4 x}{4}$

27. $\int (1+3\sin^2 x)^2 \sin x \cos x\,dx = \frac{(1+3\sin^2 x)^3}{18}$

28. $\int \frac{\sin x\,dx}{\cos x} = \log\,(\sec x)$

29. $\int (a^2+b^2\sin^2 x)^{\frac{1}{2}} \sin x \cos x\,dx = \frac{(a^2+b^2\sin^2 x)^{\frac{3}{2}}}{3b^2}$

30. $\int \sin\,(nx)\cos^3(nx)\,dx = -\,\frac{\cos^4(nx)}{4n}$

31. $\int \sin^n(nx)\cos\,(nx)\,dx = \frac{\sin^{n+1}(nx)}{n\,(n+1)}$

32. $\int \sin^5\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)dx = \frac{1}{3}\sin^6\left(\frac{x}{2}\right)$

33. $\int \frac{x\,dx}{\cos^2(x^2)} = \frac{1}{2}\tan\,(x^2)$

34. $\int \frac{\tan^2 x\,dx}{\cos^2 x} = \frac{1}{3}\tan^3 x$

35. $\int \sec^4 x \sin x\,dx = \frac{1}{3}\sec^3 x$

36. $\int \operatorname{cosec}^3 x \cos x \, dx = -\frac{1}{2} \operatorname{cosec}^2 x$

37. $\int \frac{\sec^2 x}{\tan x} \, dx = \log (\tan x)$

38. $\int \operatorname{vers} x \sin x \, dx = \frac{1}{2} \operatorname{vers}^2 x$

39. $\int (1 - \cos^2 x) \sin x \, dx = \frac{\cos^3 x}{3} - \cos x$

40. $\int (\sin^3 x - \cos^3 x) \sin x \cos x \, dx = \frac{\sin^5 x + \cos^5 x}{5}$

41. $\int \frac{\cos x \, dx}{(a^2 - 3 \sin x)^{\frac{1}{2}}} = -\frac{2}{3} (a^2 - 3 \sin x)^{\frac{1}{2}}$

42. $\int 6 \sec (4x) \tan (4x) \, dx = \frac{3}{2} \sec (4x)$

43. $\int 2 \sin (a + 3x) \, dx = -\frac{2}{3} \cos (a + 3x)$

44. $\int \frac{3 \log^2 x \, dx}{x} = (\log x)^3$

45. $\int \frac{\cos \left(\frac{x}{2}\right) dx}{\sin^2 \left(\frac{x}{2}\right)} = -2 \operatorname{cosec} \left(\frac{x}{2}\right)$

46. $\int \frac{\sin x \, dx}{\cos^5 x} = \frac{1}{4} \sec^4 x$

47. $\int \frac{\cos x \, dx}{a - \sin x} = \log \frac{1}{a - \sin x}$

48. $\int \frac{\sin x \, dx}{(a - \cos x)^2} = -\frac{1}{a - \cos x}$

49. $\int \frac{\sin^2 x \cos x \, dx}{(a^2 - b \sin^3 x)^{\frac{1}{2}}} = -\frac{2}{3b} (a^2 - b \sin^3 x)^{\frac{1}{2}}$

50. $\int a^{3x} \, dx = \frac{a^{3x}}{3 \log a}$

51. $\int e^{2x^2} x \, dx = \frac{1}{4} e^{2x^2}$

52. $\int e^{(x^3 - x^2)} (3x^2 - 2x) \, dx = e^{(x^3 - x^2)}$

53. $\int a^{\frac{x}{a}} \, dx = \frac{a \, a^{\frac{x}{a}}}{\log a}$

54. $\int m \, e^{-nx} \, dx = -\frac{m}{n} e^{-nx}$

55. $\int (e^x - e^{-x})\,dx = e^x + e^{-x}$

56. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \log(e^x + e^{-x})$

57. $\int e^{\sin x} \cos x\,dx = e^{\sin x}$

58. $\int e^{-\cos^2 x} \sin x \cos x\,dx = \frac{1}{2} e^{-\cos^2 x}$

59. $\int (e^{2x} - e^{-2x})\,dx = \frac{1}{2}(e^{2x} + e^{-2x})$

60. $\int e^{\sec x} \sec^2 x \sin x\,dx = e^{\sec x}$

61. $\int \frac{e^{-\tan^{-1}\frac{x}{a}}\,dx}{a^2 + x^2} = -\frac{e^{-\tan^{-1}\frac{x}{a}}}{a}$

62. $\int (1 - a^x)^2 a^x\,dx = -\frac{(1 - a^x)^3}{3 \log a}$

63. $\int (3x^2 - e^x)^{\frac{3}{2}} (6x - e^x)\,dx = \frac{2(3x^2 - e^x)^{\frac{5}{2}}}{5}$

64. $\int \frac{e^{\log x}\,dx}{x} = x$

65. $\int a^{\cos(3x)} \sin(3x)\,dx = -\frac{a^{\cos(3x)}}{3 \log a}$

66. $\int a^{(a+bx)}\,dx = \frac{a^{(a+bx)}}{b \log a}$

67. $\int \frac{dx}{(a^2 - b^2x^2)^{\frac{1}{2}}} = \frac{1}{b} \sin^{-1}\left(\frac{bx}{a}\right)$

68. $\int \frac{dx}{\{a - (x+b)^2\}^{\frac{1}{2}}} = \sin^{-1}\left(\frac{x+b}{\sqrt{a}}\right)$

69. $\int \frac{dx}{1 + 5x^2} = \frac{1}{\sqrt{5}} \tan^{-1}(x\sqrt{5})$

70. $\int \frac{2dx}{x(3x^2 - 5)^{\frac{1}{2}}} = \frac{2}{\sqrt{5}} \sec^{-1}\left\{x\sqrt{\frac{3}{5}}\right\}$

71. $\int -\frac{2x^{-1}\,dx}{(14x^2 - 3)^{\frac{1}{2}}} = \frac{2}{\sqrt{3}} \operatorname{cosec}^{-1}\left\{x\sqrt{\frac{14}{3}}\right\}$

72. $\int \frac{x^2\,dx}{x^6 + 1} = \frac{1}{3} \tan^{-1} x^3$

73. $\int \frac{x^2\,dx}{\sqrt{(1 - x^6)}} = \frac{1}{3} \sin^{-1} x^3$

36. $\int \operatorname{cosec}^3 x \cos x \, dx = -\frac{1}{2} \operatorname{cosec}^2 x$

37. $\int \frac{\sec^2 x}{\tan x} dx = \log (\tan x)$

38. $\int \operatorname{vers} x \sin x \, dx = \frac{1}{2} \operatorname{vers}^2 x$

39. $\int (1 - \cos^2 x) \sin x \, dx = \frac{\cos^3 x}{3} - \cos x$

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46. $\int \frac{\sin x \, dx}{\cos^5 x} = \frac{1}{4} \sec^4 x$

47. $\int \frac{\cos x \, dx}{a - \sin x} = \log \frac{1}{a - \sin x}$

48. $\int \frac{\sin x \, dx}{(a - \cos x)^2} = -\frac{1}{a - \cos x}$

49. $\int \frac{\sin^2 x \cos x \, dx}{(a^2 - b \sin^3 x)^{\frac{1}{2}}} = -\frac{2}{3b} (a^2 - b \sin^3 x)^{\frac{1}{2}}$

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53. $\int a^{\frac{x}{a}} \, dx = \frac{a \, a^{\frac{x}{a}}}{\log a}$

54. $\int m e^{-nx} \, dx = -\frac{m}{n} e^{-nx}$

55. $\int (e^x - e^{-x})\,dx = e^x + e^{-x}$

56. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \log(e^x + e^{-x})$

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58. $\int e^{-\cos^2 x}\sin x\cos x\,dx = \frac{1}{2}\,e^{-\cos^2 x}$

59. $\int (e^{2x} - e^{-2x})\,dx = \frac{1}{2}\,(e^{2x} + e^{-2x})$

60. $\int e^{\sec x}\sec^2 x\sin x\,dx = e^{\sec x}$

61. $\int \frac{e^{-\tan^{-1}\frac{x}{a}}\,dx}{a^2 + x^2} = -\frac{e^{-\tan^{-1}\frac{x}{a}}}{a}$

62. $\int (1 - a^x)^2 a^x\,dx = -\frac{(1 - a^x)^3}{3\log a}$

63. $\int (3x^2 - e^x)^{\frac{3}{2}}(6x - e^x)\,dx = \frac{2\,(3x^2 - e^x)^{\frac{5}{2}}}{5}$

64. $\int \frac{e^{\log x}\,dx}{x} = x$

65. $\int a^{\cos(3x)}\sin(3x)\,dx = -\frac{a^{\cos(3x)}}{3\log a}$

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68. $\int \frac{dx}{\{a - (x+b)^2\}^{\frac{1}{2}}} = \sin^{-1}\left(\frac{x+b}{\sqrt{a}}\right)$

69. $\int \frac{dx}{1 + 5x^2} = \frac{1}{\sqrt{5}}\tan^{-1}(x\sqrt{5})$

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73. $\int \frac{x^2\,dx}{\sqrt{(1 - x^6)}} = \frac{1}{3}\sin^{-1}x^3$

74. $$\int \frac{x\,dx}{(a^2 - b^2x^4)^{\frac{1}{2}}} = \frac{1}{2b} \sin^{-1}\left(\frac{bx^2}{a}\right)$$

75. $$\int \frac{dx}{(ax - x^2)^{\frac{1}{2}}} = \text{vers}^{-1}\left\{\frac{2x}{a}\right\}$$

76. $$\int \frac{dx}{(4ax - x^2)^{\frac{1}{2}}} = \text{vers}^{-1}\left\{\frac{x}{2a}\right\}$$

77. $$\int \frac{dx}{(2ax - b^2x^2)^{\frac{1}{2}}} = \frac{1}{b}\,\text{vers}^{-1}\left\{\frac{b^2x}{a}\right\}$$

78. $$\int \frac{3dx}{(b^2x - 4ax^2)^{\frac{1}{2}}} = \frac{3}{2\sqrt{a}}\,\text{vers}^{-1}\left\{\frac{8ax}{b^2}\right\}$$

79. $$\int \frac{dx}{(b^2 - 3x^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{3}} \sin^{-1}\left\{\frac{x\sqrt{3}}{b}\right\}$$

80. $$\int \frac{dx}{25 + 9x^2} = \frac{1}{15}\tan^{-1}\left(\frac{3x}{5}\right)$$

81. $$\int \frac{a\,b\,dx}{x\,(a^2x^2 - b^2)^{\frac{1}{2}}} = a \sec^{-1}\left(\frac{ax}{b}\right)$$

82. $$\int \frac{x^2\,dx}{x^3\,(x^6 - a^2)} = \frac{1}{3a} \sec^{-1}\left(\frac{x^3}{a}\right)$$

83. $$\int \frac{x\,dx}{a^2 + b^2x^4} = \frac{1}{2ab}\tan^{-1}\left(\frac{bx^2}{a}\right)$$

84. $$\int \frac{x^2\,dx}{a^3 + b^2x^6} = \frac{1}{3a^{\frac{3}{2}}\,b} \tan^{-1}\left\{\frac{bx^3}{\sqrt{a^3}}\right\}$$

85. $$\int \frac{dx}{\tan x} = \log \sin x$$

86. $$\int \frac{dx}{e^x + e^{-x}} = \tan^{-1} e^x$$

87. $$\int \frac{dx}{x \log x} = \log(\log x)$$

88. $$\int \frac{\cos x\,dx}{(a^2 - \sin^2 x)^{\frac{1}{2}}} = \sin^{-1}\left\{\frac{\sin x}{a}\right\}$$

89. $$\int \frac{dx}{(7ax - 9x^2)^{\frac{1}{2}}} = \tfrac{1}{3}\,\text{vers}^{-1}\left(\frac{18x}{7a}\right)$$

90. $$\int \frac{3adx}{x\,(9x^2 - 4)^{\frac{1}{2}}} = \frac{3a}{2} \sec^{-1}\left(\frac{3x}{2}\right)$$

91. $$\int \frac{\sin x\,dx}{(1 + \cos x)^6} = \frac{1}{5\,(1 + \cos x)^5}$$

92. $\int (\cos x + \sin x)^3 (\cos x - \sin x)\, dx = \frac{(\cos x + \sin x)^4}{4}$

93. $\int \frac{5x^3\, dx}{3x^4 + 7} = \frac{5}{12} \log (3x^4 + 7)$

94. $\int \frac{dx}{\sqrt{(a - bx^2)}} = \frac{1}{\sqrt{b}} \sin^{-1} \left\{ x\sqrt{\frac{b}{a}} \right\}$

95. $\int \frac{dx}{\sin 2x} = \frac{1}{2} \log \tan x$

96. $\int \frac{dx}{x^n} = - \frac{1}{(n-1)\, x^{n-1}}$

97. $\int \left\{ \frac{dx}{a + x} - \frac{dx}{b + x} \right\} = \log \frac{a + x}{b + x}$

98. $\int \sin 2x\, dx = \sin^2 x$

99. $\int \frac{\sqrt{x}\, dx}{\{2(1 - 2x^3)\}^{\frac{1}{2}}} = \frac{1}{3} \sin^{-1} \{x\sqrt{2x}\}$

100. $\int \frac{dx}{(2x - 5x^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{5}} \operatorname{vers}^{-1} \{5x\}$

## INTEGRATION BETWEEN LIMITS.

### EXAMPLES.

### II.

1. $\int_1^2 x^5\, dx = 10.5$

2. $\int_0^2 (x^3 - x^2)\, dx = \frac{4}{3}$

3. $\int_0^{\frac{\pi}{2}} \cos x\, dx = 1$

4. $\int_{-\frac{a}{b}}^{0} (a + bx)^2\, dx = \frac{a^3}{3b}$

5. $\int_0^a \frac{dx}{\sqrt{(a^2 - x^2)}} = \frac{\pi}{2}$

6. $\int_0^a \frac{dx}{a^2 + x^2} = \frac{\pi}{4a}$

7. $\int_0^{\frac{\pi}{4}} \frac{\sin x\, dx}{\cos x} = \frac{1}{2} \log 2$

8. $\int_0^{\frac{a}{2}} \frac{dx}{(a-x)^2} = \frac{1}{a}$

9. $\int_0^{a} \frac{3dx}{ax - x^2)^{\frac{1}{2}}} = 3\pi$

10. $\int_0^{3} \frac{dx}{36 + 4x^2} = \frac{\pi}{48}$

11. $\int_{-a}^{0} \frac{x\, dx}{\sqrt{(a^2 - x^2)}} = -a$

12. $\int_1^{\infty} \left\{ \frac{1}{x^2} - \frac{1}{x^4} \right\} dx = \frac{2}{3}$

13. $\int_0^{1} \{x^6 - x^4 + x^2\}\, dx = \frac{29}{105}$

14. $\int_0^{\infty} \frac{dx}{16 + 9x^2} = \frac{\pi}{24}$

15. $\int_{-\infty}^{0} e^x\, dx = 1$

16. $\int_0^{\sqrt{\frac{\pi}{4}}} \frac{x\, dx}{\cos^2 (x^2)} = \frac{1}{2}$

17. $\int_0^{\frac{\pi}{4}} \cos^5 x \sin x\, dx = \frac{7}{48}$

18. $\int_0^{\frac{\pi}{2}} \sin^4 x \cos x\, dx = \frac{1}{5}$

19. $\int_0^{\frac{\pi}{4}} \sin (2x) \cos (2x)\, dx = \frac{1}{4}$

20. $\int_0^{a^{\frac{2}{3}}} \frac{x^2\, dx}{3x^2 + a^2} = \frac{1}{9} \log 4$

21. $\int_0^{\frac{\pi}{2}} e^{-\pi \tan \theta} \sec^2 \theta\, d\theta = -\frac{1}{\pi}$

22. $\int_0^{\frac{\pi}{4}} e^{\sin (2x)} \cos (2x)\, dx = \frac{e - 1}{2}$

23. $\int_0^{\frac{a}{2}} \frac{dx}{(a^3 - ax^2)^{\frac{3}{2}}} = \frac{\pi}{6\sqrt{a}}$

## INTEGRATION BY CHANGE OF FORM.

**10.** A quantity may become directly integrable by a mere change of form; thus,

By *expansion*,

$$\int (1 - x^2)^2\, dx = \int dx - 2\int x^2\, dx + \int x^4\, dx$$

the terms of which are directly integrable.

By *division*, as

$$\int \frac{x^3}{a - x} = \int - x^2\, dx - \int ax\, dx - a^2 \int dx - a^3 \int \frac{dx}{x - a}$$

By *separating the parts of a fraction*, as

$$\int \frac{a + bx}{a^2 + x^2}\, dx = a\int \frac{dx}{a^2 + x^2} + b\int \frac{x\, dx}{a^2 + x^2}$$

By *adding and subtracting the same quantity;* thus,

$$\int \sqrt{(a - x)}\, x\, dx = - \int \sqrt{(a - x)}\, \{(a - x) - a\}\, dx$$

$$= - \int (a - x)^{\frac{3}{2}}\, dx + a \int \sqrt{(a - x)}\, dx$$

This is equivalent to separating into parts, which have been disguised, by the cancellation of equal positive and negative terms.

By *multiplying numerator and denominator by the same quantity ;* thus,

$$\int \frac{\sqrt{(x^2 - a^2)}}{x}\, dx = \int \frac{x^2 - a^2}{x\sqrt{(x^2 - a^2)}}\, dx$$

$$= \int \frac{x\, dx}{\sqrt{(x^2 - a^2)}} - a^2 \int \frac{dx}{x\sqrt{(x^2 - a^2)}}$$

By *trigonometrical transformation*, as

$$\int \sin^3 x\, dx = \int \sin x\, (1 - \cos^2 x)\, dx$$

$$= \int \sin x - \int \cos^2 x \sin x\, dx$$

## EXAMPLES.

### III.

1. $\int_0^a (3x + 2a)^2 x^2\, dx = \frac{92}{15} a^5$

2. $\int \frac{4x^5 + 5x^3}{x - 20}\, dx = \left\{ \begin{array}{l} \frac{4x^5}{5} + 20x^4 + 535x^3 + 16050x^2 \\ + 642000x + 12840000 \log (x - 20) \end{array} \right.$

3. $\int_0^a (x+3a)^3 x^5\,dx = 9\frac{299}{504}a^9$

4. $\int \frac{3x^3+4x^2+2x+3}{x^2+1}dx$

$$= \frac{3x^2}{2} + 4x - \tfrac{1}{2}\log(x^2+1) - \tan^{-1}x$$

5. $\int \frac{a+x}{\sqrt{(a^2-x^2)}}dx = a\sin^{-1}\left(\frac{x}{a}\right) - \sqrt{(a^2-x^2)}$

6. $\int \frac{x^2\,dx}{a^2+x^2} = x - a\tan^{-1}\left(\frac{x}{a}\right)$

7. $\int \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}dx = \int \frac{a+x}{(a^2-x^2)^{\frac{1}{2}}}dx$

$$= a\sin^{-1}\left(\frac{x}{a}\right) - \sqrt{(a^2-x^2)}$$

8. $\int \frac{(2ax-x^2)^{\frac{1}{2}}}{x}dx = \int \frac{2ax-x^2}{x\sqrt{(2ax-x^2)}}dx$

$$= \int \frac{(a-x)\,dx}{\sqrt{(2ax-x^2)}} + \int \frac{a\,dx}{\sqrt{(2ax-x^2)}}$$

$$= \sqrt{(2ax-x^2)} + a\,\mathrm{vers}^{-1}\left(\frac{x}{a}\right)$$

9. $\int \frac{x^2+a}{x\sqrt{(x^2-a^2)}} = \sqrt{(x^2-a^2)} + \sec^{-1}\left(\frac{x}{a}\right)$

10. $\int \sqrt{(x+a)}\,x\,dx = \int \sqrt{(x+a)}\,\{(x+a)-a\}\,dx$

$$= \int (x+a)^{\frac{3}{2}}dx - \int a\,(x+a)^{\frac{1}{2}}dx$$

$$= \frac{2(x+a)^{\frac{5}{2}}}{5} - \frac{2a\,(x+a)^{\frac{3}{2}}}{3}$$

11. $\int \frac{dx}{(a-bx-x^2)^{\frac{1}{2}}} = \int \frac{dx}{\left[a+\frac{b^2}{4}-\left(\frac{b}{2}+x\right)^2\right]^{\frac{1}{2}}}$

$$= \sin^{-1}\frac{b+2x}{\sqrt{(4a+b^2)}}$$

12. $\int_0^{\frac{\pi}{2}} \cos^5 x\,dx = \frac{8}{15}$

13. $\int \sin^3\theta\cos^3\theta\,d\theta = \frac{\cos^6\theta}{6} - \frac{\cos^4\theta}{4}$

14. $\int \sec^4\theta\,d\theta = \tan\theta + \frac{\tan^3\theta}{3}$

15. $\int \sec^4\theta \tan\theta \, d\theta = \frac{\tan^2\theta}{2} + \frac{\tan^4\theta}{4}$

16. $\int \sec^6\theta \, d\theta = \int (\tan^2\theta + 1)^2 \sec^2\theta \, d\theta$

$$= \tan\theta + \frac{2\tan^3\theta}{3} + \frac{\tan^5\theta}{5}$$

17. $\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$$= \int \frac{\sec^2\frac{x}{2}\cdot\frac{dx}{2}}{\tan\frac{x}{2}}$$

$$= \log\tan\left(\frac{x}{2}\right)$$

18. $\int_0^{\frac{\pi}{4}} \tan^7 x \, dx = \frac{5}{12} + \frac{1}{2}\log\left(\frac{1}{2}\right)$

19. $\int \tan^2 x \, dx = \tan x - x$

20. $\int \tan^3 x \, dx = \frac{\tan^2 x}{2} + \log\cos x$

21. $\int \frac{dx}{\sin^2 x \cos^2 x} = -2\cot(2x)$

$$= \tan x - \cot x$$

22. $\int \frac{dx}{\sqrt{(x+a)} + \sqrt{(x+b)}} = \int \frac{\sqrt{(x+a)} - \sqrt{(x+b)}}{a-b} dx$

$$= \frac{2}{3}\,\frac{(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}}{a-b}$$

23. $\int \frac{dx}{1+\sin x} = \tan x - \sec x$

24. $\int \frac{dx}{1+\cos x} = \operatorname{cosec} x - \cot x$

25. $\int \frac{dx}{x(a+bx^3)} = \frac{1}{a}\int \frac{(a+bx^3) - bx^3}{x(a+bx^3)} dx$

$$= \frac{1}{3a}\log\left\{\frac{x^3}{a+bx^3}\right\}$$

26. $\int \frac{dx}{x^4(a+bx^3)} = \frac{b}{3a^2}\log\left\{\frac{a+bx^3}{x^3}\right\} - \frac{1}{3ax^3}$

27. $\int \frac{dx}{x(a+bx^4)} = \frac{1}{4a}\log\left\{\frac{x^4}{a+bx^4}\right\}$

28. $\int \frac{(x+1)\,dx}{x^2+x+1} = \frac{1}{2}\log(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$

29. $\int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left\{\frac{\pi}{2}-x\right\}} = \log\tan\left\{\frac{\pi}{4}+\frac{x}{2}\right\}$

30. $\int \frac{dx}{\sin x+\cos x} = \int \frac{dx}{\sin x + \sin(90-x)}$

$$= \int \frac{dx}{\sqrt{2}\cos\left\{x-\frac{\pi}{4}\right\}}$$

$$= \frac{1}{\sqrt{2}}\log\tan\left\{\frac{x}{2}+\frac{\pi}{8}\right\}$$

## INTEGRATION OF RATIONAL FRACTIONS.

**11.** Any fraction whose terms are rational functions of $x$ can be integrated, provided the denominator can be resolved into factors of the first and second degrees.

If the numerator is of a degree equal to or higher than that of the denominator, the fraction must be reduced by division to the form of a mixed quantity. The entire part will be directly integrable. We, therefore, have to consider those fractions only whose denominators are of a higher degree than their numerators. The denominator is supposed to be resolved into factors of the first degree, and quadratic factors which cannot be resolved into real factors of the first degree. The latter can be put in the form

$$(ax+b)^2+c^2$$

The following examples will serve to explain the method of decomposing into rational fractions, and of integrating the results.

### EXAMPLES.

### IV.

1. $\int \frac{x-1}{x^2+6x+8}\,dx$

To decompose, put

$$\frac{x-1}{(x+2)(x+4)} = \frac{A}{x+2} + \frac{B}{x+4}$$

whence

$$x - 1 = A\,(x + 4) + B\,(x + 2)$$

which must be true for all values of $x$. Then, to determine $A$, let $x = -2$, which will make the term containing $B$ disappear; thus,

$$-3 = A\,(2)$$

or

$$A = -\tfrac{3}{2}$$

To determine $B$, let $x = -4$:

$$-5 = B\,(-2)$$

or

$$B = \tfrac{5}{2}$$

Then

$$\int \frac{x - 1}{x^2 + 6x + 8}\,dx = \tfrac{5}{2}\int \frac{dx}{x + 4} - \tfrac{3}{2}\int \frac{dx}{x + 2}$$

$$= \log \frac{(x + 4)^{\frac{5}{2}}}{(x + 2)^{\frac{3}{2}}}$$

2. $\displaystyle\int \frac{9x^2 + 9x - 128}{x^3 - 5x^2 + 3x + 9}\,dx$

Put

$$\frac{9x^2 + 9x - 128}{(x - 3)^2\,(x + 1)} = \frac{A}{(x - 3)^2} + \frac{B}{x - 3} + \frac{C}{x + 1} \cdots\cdots (1)$$

whence

$$9x^2 + 9x - 128 = A\,(x + 1) + B\,(x + 1)\,(x - 3) + C\,(x - 3)^2$$

Put

$$x = 3;\ \text{then} -20 = A\,(4) \qquad \therefore A = -5$$
$$x = -1;\ \text{then} -128 = C\,(16) \qquad \therefore C = -8$$
$$x = 0;\ \text{then} -128 = -5 - 3B - 72 \therefore B = +17$$

Substituting in (1),

$$\frac{9x^2 + 9x - 128}{(x - 3)^2\,(x + 1)}\,dx = -\frac{5dx}{(x - 3)^2} + \frac{17dx}{x - 3} - \frac{8dx}{x + 1}$$

3. $\displaystyle\int \frac{x^2\,dx}{x^4 + x^2 - 2}$

$$\frac{x^2}{x^4 + x^2 - 2} = \frac{Ax + B}{x^2 + 2} + \frac{C}{x + 1} + \frac{D}{x - 1}$$

$$x^2 = Ax\,(x^2 - 1) + B\,(x^2 - 1) + C\,(x^2 + 2)\,(x - 1) + D\,(x^2 + 2)\,(x + 1)$$

Put

$$x = -1;\ \text{then}\ \ 1 = C\,(-6) \therefore C = -\tfrac{1}{6}$$
$$x = 1;\ \text{then}\ \ 1 = D\,(6) \quad \therefore D = +\tfrac{1}{6}$$
$$x^2 = -2;\ \text{then} -2 = Ax\,(-3) + B\,(-3)$$

or

$$2 = 3Ax + 3B$$

In which $x$ is imaginary; equating the coefficients of the real and of the imaginary parts, we have

$$A = 0$$

and

$$B = \tfrac{2}{3}$$

hence

$$\int \frac{x^2\,dx}{x^4 + x^2 - 2} = \int \frac{\frac{2}{3}\,dx}{x^2 + 2} + \int \frac{\frac{1}{6}\,dx}{x - 1} - \int \frac{\frac{1}{6}\,dx}{x + 1}$$

$$= \tfrac{1}{6} \log \left\{ \frac{x-1}{x+1} \right\} + \frac{\sqrt{2}}{3} \tan^{-1} \left\{ \frac{x}{\sqrt{2}} \right\}$$

4. $\displaystyle\int \frac{x^2 - x + 2}{x^4 - 5x^2 + 4}\,dx = \log \left\{ \frac{(x+1)^2 (x-2)}{(x-1)(x+2)^2} \right\}^{\frac{1}{3}}$

5. $\displaystyle\int \frac{x^2 - 1}{x^2 - 4}\,dx = x + \log \left\{ \frac{x-2}{x+2} \right\}^{\frac{3}{4}}$

6. $\displaystyle\int \frac{x^3\,dx}{x^2 + 7x + 12} = \frac{x^2}{2} - 7x + 64 \log(x+4) - 27 \log(x+3)$

7. $\displaystyle\int \frac{x^2 - 3x + 3}{(x^2 - 3x + 2)} = x + \log \left( \frac{x-2}{x-1} \right)$

8. $\displaystyle\int \frac{3x - 1}{x^3 - x^2 - 2x}\,dx = \tfrac{1}{2} \log x + \tfrac{5}{6} \log(x-2) - \tfrac{4}{3} \log(x+1)$

9. $\displaystyle\int \frac{x\,dx}{x^2 + 6x + 8} = \log \left\{ \frac{(x+4)^2}{(x+2)} \right\}$

10. $\displaystyle\int \frac{5x^3 + 1}{x^2 - 3x + 2}\,dx$

$$= \frac{5x^2}{2} + 15x - 6 \log(x+1) + 41 \log(x-2)$$

11. $\displaystyle\int \frac{dx}{x^3 - x^2 - x + 1} = \log \left( \frac{x+1}{x-1} \right)^{\frac{1}{4}} - \frac{1}{2(x-1)}$

12. $\displaystyle\int \frac{2x^3 + 7x^2 + 6x + 2}{x^4 + 3x^3 + 2x^2}\,dx = \log \left[ x(x+1) \sqrt{\frac{x}{x+2}} \right] - \frac{1}{x}$

13. $\displaystyle\int \frac{x^2\,dx}{x^3 + 5x^2 + 8x + 4} = \frac{4}{x+2} + \log(x+1)$

14. $\displaystyle\int \frac{x\,dx}{(x+2)(x+3)^2} = -\frac{3}{x+3} + \log \left( \frac{x+3}{x+2} \right)^2$

15. $\displaystyle\int \frac{x^2\,dx}{1 - x^6} = \tfrac{1}{6} \log \left( \frac{1 + x^3}{1 - x^3} \right)$

16. $\displaystyle\int \frac{2x^2 - 3a^2}{x^4 - a^4} = \frac{5}{2a} \tan^{-1} \frac{x}{a} - \frac{1}{4a} \log \left( \frac{x-a}{x+a} \right)$

17. $\displaystyle\int \frac{dx}{(x^2 + 1)(x^2 + x + 1)}$

$$= \tfrac{1}{2}\log\left(\frac{x^2+x+1}{x^2+1}\right) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$$

18. $$\int\frac{x^2-1}{x^4+x^2+1}\,dx = \tfrac{1}{2}\log\left\{\frac{x^2-x+1}{x^2+x+1}\right\}$$

19. $$\int\frac{dx}{(x^2+a^2)(x+b)}$$
$$= \frac{1}{b^2+a^2}\left\{\log\left(\frac{x+b}{\sqrt{(x^2+a^2)}}\right) + \frac{b}{a}\tan^{-1}\frac{x}{a}\right\}$$

20. $$\int\frac{dx}{x(1+x+x^2+x^3)}$$
$$= \log x - \tfrac{1}{2}\log(1+x) - \tfrac{1}{4}\log(1+x^2) - \tfrac{1}{2}\tan^{-1}x$$

21. $$\int\frac{dx}{(x-1)^2(x^2+1)^2} = -\frac{1}{4(x-1)} - \tfrac{1}{2}\log(x-1)$$
$$+ \tfrac{1}{4}\tan^{-1}x - \frac{1}{4(x^2+1)} + \tfrac{1}{4}\log(x^2+1)$$

22. $$\int\frac{x^2-1}{x^4+1}dx = \frac{1}{2\sqrt{2}}\log\left\{\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right\}$$

23. $$\int\frac{x^2\,dx}{x^4+1} = \frac{1}{4\sqrt{2}}\log\left\{\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right\}$$
$$+\frac{1}{2\sqrt{2}}\{\tan^{-1}(x\sqrt{2}+1) + \tan^{-1}(x\sqrt{2}-1)\}$$

24. $$\int\frac{x^3\,dx}{x^6+1} = \tfrac{1}{12}\log(x^4-x^2+1) - \tfrac{1}{6}\log(x^2+1)$$
$$+\frac{1}{2\sqrt{3}}\{\tan^{-1}(2x-\sqrt{3}) - \tan^{-1}(2x+\sqrt{3})\}$$

25. $$\int\frac{x\,dx}{(1+x)(1+2x)^2(1+x^2)} = \frac{2}{5}\cdot\frac{1}{1+2x} - \tfrac{1}{2}\log(1+x)$$
$$-\tfrac{7}{100}\log(1+x^2) + \tfrac{16}{25}\log(1+2x) + \tfrac{1}{50}\tan^{-1}x$$

## INTEGRATION BY SUBSTITUTION.

**12.** In the preceding examples, we have been able to see at once that, by an easily-suggested transformation, the given differential, or its several parts, will take forms in which a function of $x$ and its differential take the place of $x$ and $dx$ in one or another of the fundamental integrals. In more complex examples, when this is not the case, it may yet happen that a function of $x$ may occur in the quantity in such a way that, by the substitution of a single letter (or some other function of a new variable) for that variable, an expression of the whole given quantity in terms of the new variable will be of an integrable form. In this case, we cannot usually foresee the new form of expression, but can only be guided by general views of simplification.

**13.** The simplest examples are those in which a simple expression in $x$ occurs under the radical sign, while a rational expression in $x$ is another factor; thus,

$$\int \sqrt{(a + x)}\, a^2\, dx$$

Here, it is easily seen that, if we substitute $z$ for $a + x$, the whole expression becomes a series of powers of $z$.

Since

$$\begin{aligned} a + x &= z \\ x &= z - a \\ dx &= dz \end{aligned}$$

hence

$$\begin{aligned} \int \sqrt{(a + x)}\, x^2\, dx &= \int \sqrt{z}\, (z - a)^2\, dz \\ &= \int z^{\frac{5}{2}}\, dz - 2a \int z^{\frac{3}{2}}\, dz + a^2 \int z^{\frac{1}{2}}\, dz \\ &= \tfrac{2}{7} z^{\frac{7}{2}} - \tfrac{4}{5}\, a\, z^{\frac{5}{2}} + \tfrac{2}{3}\, a^2\, z^{\frac{3}{2}} \\ &= \tfrac{2}{7}\, (a + x)^{\frac{7}{2}} - \tfrac{4}{5}\, a\, (a + x)^{\frac{5}{2}} + \tfrac{2}{3}\, a^2\, (a + x)^{\frac{3}{2}} \end{aligned}$$

As another example, take the case where a radical occurs in the denominator; as,

$$\int \frac{x\, dx}{1 + \sqrt{x}}$$

Let

$$\begin{aligned} z &= \sqrt{x} \\ x &= z^2 \\ dx &= 2z\, dz \end{aligned}$$

$$\begin{aligned} \int \frac{x\, dx}{1 + \sqrt{x}} &= \int \frac{z^2\, 2z\, dz}{1 + z} \\ &= 2 \left[ \int z^2\, dz - \int z\, dz + \int dz - \int \frac{dz}{1 + z} \right] \\ &= \tfrac{2}{3} z^3 - z^2 + 2z - 2 \log(1 + z) \\ &= \tfrac{2}{3} x^{\frac{3}{2}} - x + 2\sqrt{x} - 2 \log(1 + \sqrt{x}) \end{aligned}$$

Again, where the denominator contains a power of $x$, together with a factor containing $x$, it is useful to substitute, for $x$, $\frac{1}{z}$; whence $z = \frac{1}{x}$. This will tend to throw the power of the variable into the numerator. For instance,

$$\int \frac{dx}{x^3(x+1)}$$

Assume

$$x = \frac{1}{z}$$

$$dx = -\frac{dz}{z^2}$$

$$\int \frac{dx}{x^3(x+1)} = -\int \frac{\frac{dz}{z^2}}{\frac{1}{z^3}\cdot\frac{1+z}{z}} = -\int \frac{z^2\,dz}{1+z}$$

$$= -\frac{z^2}{2} + z - \log(1+z)$$

$$= -\frac{1}{2x^2} + \frac{1}{x} - \log\left\{\frac{x+1}{x}\right\}$$

**14.** A still more useful substitution than any of those mentioned is that of a trigonometric function of a new variable for the independent variable, where other functions of the same angle are seen to be equal to other factors (principally radicals) in the given differential; but this substitution is, in general, only useful where the whole differential can be expressed in trigonometric functions. Thus, in

$$\int \frac{dx}{x^2\sqrt{(1+x^2)}}$$

if we put $x = \tan\theta$, the radical will become $\sec\theta$, and

$$\int \frac{dx}{x^2\sqrt{(1+x^2)}} = \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sec\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta}$$

$$= -\frac{\sqrt{(1+x^2)}}{x}$$

The facility with which trigonometrical transformations are made renders this the most successful way of dealing with radicals.

**15.** Another substitution is that of a single letter, $z$, for an exponential function of $x$; $x$ will then be a logarithmic function of $z$, and $dx$ will be algebraic. Thus,

$$\int \frac{dx}{1+e^x}$$

Let

$$z = e^x$$

$$x = \log z$$

$$dx = \frac{dz}{z}$$

$$\int \frac{dx}{1+e^x} = \int \frac{\frac{dz}{z}}{1+z} = \int \frac{dz}{z(1+z)} = \int \frac{dz}{z} - \int \frac{dz}{1+z}$$

$$= \log \left\{ \frac{z}{1+z} \right\}$$

$$= \log \left\{ \frac{e^x}{1+e^x} \right\}$$

**16.** When the method of substitution is employed for obtaining the value of any integral between limits, if, for the limits of the old variable, we substitute corresponding values of the new variable, it will not be necessary to reintroduce the old variable in the final result, the computation being made from the new form of the integral; thus,

$$\int \frac{dx}{x^2 \sqrt{(1-x^2)}}$$

Put

$$x = \sin \theta$$

then

$$\sqrt{(1-x^2)} = \cos \theta$$

$$dx = \cos \theta \, d\theta$$

$$\int_{\frac{1}{2}}^{1} \frac{dx}{x^2 \sqrt{(1-x^2)}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos \theta \, d\theta}{\sin^2 \theta \cos \theta} = \Big[ -\cot \theta \Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \sqrt{3}$$

### EXAMPLES.

### V.

1. $\int \frac{dx}{\sqrt{(x^2+a^2)}} = \log \{x + \sqrt{(x^2+a^2)}\}$

2. $\int \frac{dx}{\sqrt{(x^2-a^2)}} = \log \{x + \sqrt{(x^2-a^2)}\}$

3. $\int \frac{dx}{x^2 \sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{x}$

4. $\int (3x+a)^{\frac{1}{2}} x^2 \, dx$

$$= \frac{1}{27} \left\{ \frac{2}{7} (3x+a)^{\frac{7}{2}} - \frac{4a}{5} (3x+a)^{\frac{5}{2}} + \frac{2a^2}{3} (3x+a)^{\frac{3}{2}} \right\}$$

5. $\int \frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{\left(\frac{2x^2}{1-x^2}\right)}$

6. $\int \frac{x\,dx}{\sqrt{(x+a)}} = \frac{2}{3}(x+a)^{\frac{3}{2}} - 2a(x+a)^{\frac{1}{2}}$

7. $\int \frac{x^2\,dx}{\sqrt{(x+a)}} = \frac{2}{5}(x+a)^{\frac{5}{2}} - \frac{4a}{3}(x+a)^{\frac{3}{2}} + 2a^2(x+a)^{\frac{1}{2}}$

8. $\int (a+bx)^{\frac{3}{2}}\,x\,dx = \frac{1}{b^2}\left\{\frac{2}{7}(a+bx)^{\frac{7}{2}} - \frac{2a}{5}(a+bx)^{\frac{5}{2}}\right\}$

9. $\int \frac{x^3\,dx}{(x+a)^2}$

$$= \frac{(x+a)^2}{2} - 3a(x+a) + 3a^2\log(x+a) + \frac{a^3}{x+a}$$

10. $\int \frac{dx}{ax^4+bx^2} = \frac{1}{b}\left\{\sqrt{\left(\frac{a}{b}\right)}\tan^{-1}\sqrt{\left(\frac{b}{ax^2}\right)} - \frac{1}{x}\right\}$

11. $\int \frac{x^2\sqrt{x}\,dx}{1+x} = \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 2\tan^{-1}\sqrt{x}$

12. $\int \frac{dx}{(a+bx^2)^{\frac{3}{2}}} = \frac{x}{a(a+bx^2)^{\frac{1}{2}}}$

13. $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{(a^2-x^2)}}$

14. $\int_0^a \sqrt{(a^2-x^2)}\,dx = \frac{\pi a^2}{4}$

## INTEGRATION BY PARTS.

**17.** Considering integration as the process of discovering from what function the given differential was derived, *substitution* is a method of discovering a function of a function from its differential. We now give a method of discovering the integral when the given differential is of a form which may arise from the differentiation of the product of two functions.

Let $u$ and $v$ represent two functions of $x$; then

$$d(uv) = u\,dv + v\,du$$

or

$$uv = \int u\,dv + \int v\,du$$

Two integrals of the form

$$\int u\,dv + \int v\,du$$

will rarely occur in connection; but, by transposition,

$$\int u\,dv = uv - \int v\,du$$

so that we can make the possibility of integrating $u\,dv$ depend upon the possibility of integrating $v\,du$. The application of this formula is called *integration by parts*. It is useful whenever, by separating the given differential into two factors ($u$ and $dv$), we can integrate the differential factor (*i. e.*, find $v$), and, multiplying that by the differential of the other factor ($du$), obtain a simpler integral. Thus, given

$$\int x \cos x\, dx$$

here, taking $x$ for $u$, and $\cos x\, dx$ for $dv$, since $x$ is not complicated by differentiation, nor $\cos x\, dx$ by integration, the formula is of use.

$$\begin{aligned}\int x \cos x\, dx &= x \sin x - \int \sin x\, dx \\ &= x \sin x + \cos x\end{aligned}$$

which may be verified by differentiation.

**18.** Again, given

$$\int \sin^{-1} x\, dx$$

if we make

$$u = \sin^{-1} x$$

and

$$dx = dv$$

then $v\,du$ will be algebraic and integrable; thus,

$$\begin{aligned}\int \sin^{-1} x\, dx &= x \sin^{-1} x - \int \frac{x\, dx}{\surd(1 - x^2)} \\ &= x \sin^{-1} x + \surd(1 - x^2)\end{aligned}$$

It may require several applications of this method to make the result directly integrable. Thus,

$$\int \theta^3 \sin \theta\, d\theta$$

Put

$$\theta^3 = u$$

then

$$v = -\cos \theta$$
$$du = 3\theta^2\, d\theta$$

$$\int \theta^3 \sin \theta\, d\theta = -\theta^3 \cos \theta + 3 \int \theta^2 \cos \theta\, d\theta$$

(reducing the last integral by $u = \theta^2$, $dv = \cos \theta\, d\theta$)

$$= -\theta^3 \cos \theta + 3 \left[ \theta^2 \sin \theta - 2 \int \theta \sin \theta\, d\theta \right]$$

(reducing again by $u = \theta, dv = \sin\theta\, d\theta$)

$$\int \theta^3 \sin\theta\, d\theta = -\theta^3\cos\theta + 3\theta^2\sin\theta - 6\left[-\theta\cos\theta + \int\cos\theta\, d\theta\right]$$

$$= -\theta^3\cos\theta + 3\,\theta^2\sin\theta + 6\,\theta\cos\theta - 6\sin\theta$$

**19.** Integration by parts is sometimes available, where the new integral is not more simple than the original one, but may be further reduced (by substitution, or further application of integration by parts, &c.) to a form involving the original integral, which can then be transposed to the first member. Thus,

$$\int \cos^2 x\, dx = \sin x \cos x + \int \sin^2 x\, dx$$

If we now continued to integrate by parts, we should produce the equation

$$\int \cos^2 x\, dx = \sin x \cos x - \sin x \cos x + \int \cos^2 x\, dx$$

reducing to

$$0 = 0$$

If, however, we observe that

$$\sin^2 x = 1 - \cos^2 x$$

we have

$$\int \cos^2 x\, dx = \sin x \cos x + \int dx - \int \cos^2 x\, dx$$

which gives

$$\int \cos^2 x\, dx = \frac{x + \sin x \cos x}{2}$$

### EXAMPLES.

### VI.

1. $\int x \log x\, dx = \frac{x^2 \log x}{2} - \frac{x^2}{4}$

2. $\int x^2 \sin x\, dx = 2x \sin x + 2\cos x - x^2 \cos x$

3. $\int \theta \sin\theta\, d\theta = \sin\theta - \theta\cos\theta$

4. $\int x \tan^{-1} x\, dx = \frac{1 + x^2}{2}\tan^{-1} x - \frac{x}{2}$

5. $\int (1 - \cos x)^2\, dx = \frac{3x}{2} - 2\sin x + \frac{\sin 2x}{4}$

6. $\int x^n \log x \, dx = \frac{x^{n+1}}{n+1} \left\{ \log x - \frac{1}{n+1} \right\}$

7. $\int \frac{\log(\log x)\,dx}{x} = \log x \log(\log x) - \log x$

8. $\int \sin^4 x \, dx = \frac{3x - \sin x \cos x\,(2\sin^2 x + 3)}{8}$

9. $\int \sin^6 \theta \, d\theta = \frac{15\theta - \sin\theta\cos\theta\,(8\sin^4\theta + 10\sin^2\theta + 15)}{48}$

10. $\int \cos^4 \theta \, d\theta = \frac{3\theta + \sin\theta\cos\theta\,(2\cos^2\theta + 3)}{8}$

11. $\int \frac{\sin^{-1} x \, dx}{(1-x^2)^{\frac{3}{2}}} = \frac{x \sin^{-1} x}{(1-x^2)^{\frac{1}{2}}} + \log \sqrt{(1-x^2)}$

12. $\int \cos^6 \theta \, d\theta = \frac{15\theta + \sin\theta\cos\theta\,(8\cos^4\theta + 10\cos^2\theta + 15)}{48}$

13. $\int x\, e^{ax} \, dx = e^{ax} \left\{ \frac{x}{a} - \frac{1}{a^2} \right\}$

14. $\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{(1-x^2)}$

15. $\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \log(1+x^2)$

16. $\int \frac{x^2\,dx}{x^2+1} \tan^{-1} x = x \tan^{-1} x - \frac{1}{2} \log(1+x^2) - \frac{(\tan^{-1} x)^2}{2}$

## MISCELLANEOUS EXAMPLES.

### VII.

1. $\int \sqrt{(a^2-x^2)}\,dx = x\sqrt{(a^2-x^2)} + \int \frac{x^2\,dx}{\sqrt{(a^2-x^2)}}$

$$\int \frac{a^2-x^2}{\sqrt{(a^2-x^2)}}\,dx = a^2 \sin^{-1}\frac{x}{a} - \int \frac{x^2\,dx}{\sqrt{(a^2-x^2)}}$$

by addition,

$$\int \sqrt{(a^2-x^2)}\,dx = \frac{1}{2}\left(x\sqrt{(a^2-x^2)} + a^2 \sin^{-1}\frac{x}{a}\right)$$

2. $\int \sqrt{(x^2+a^2)}\,dx = \frac{1}{2}\{x\sqrt{(x^2+a^2)} + a^2 \log[x + \sqrt{(x^2+a^2)}]\}$

3. $\int \sqrt{(x^2-a^2)}\,dx = \frac{1}{2}\{x\sqrt{(x^2-a^2)} - a^2 \log[x + \sqrt{(x^2-a^2)}]\}$

4. $$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = \frac{1}{\sqrt{c}} \int \frac{dx}{\sqrt{\left(\frac{a}{c}+\frac{bx}{c}+x^2\right)}}$$

$$= \frac{1}{\sqrt{c}} \int \frac{dx}{\sqrt{\left\{\left(x+\frac{b}{2c}\right)^2+\frac{4ac-b^2}{4c^2}\right\}}}$$

Putting

$$x+\frac{b}{2c}=z$$

this becomes, by (1) and (2) of Examples V,

$$\frac{1}{\sqrt{c}} \log\{2cx+b+2\sqrt{c}\sqrt{(a+bx+cx^2)}\}$$

where the constant quantity $\frac{1}{\sqrt{c}} \log 2c$ is omitted.

5. $$\int \sqrt{(a+bx+cx^2)}\, dx$$

$$= \sqrt{c} \int \sqrt{\left(x^2+\frac{bx}{c}+\frac{a}{c}\right)} dx$$

$$= \sqrt{c} \int \sqrt{\left\{\left(x+\frac{b}{2c}\right)^2+\frac{4ac-b^2}{4c^2}\right\}} dx$$

and, putting

$$x+\frac{b}{2c}=z$$

this becomes, by (2) and (3), of Examples VII,

$$\sqrt{c}\left\{\frac{2cx+b}{4c}\sqrt{(a+bx+cx^2)} + \frac{4ac-b^2}{8c^2}\log\left[\frac{2cx+b}{2c}+\sqrt{(a+bx+cx^2)}\right]\right\}$$

6. $$\int \frac{dx}{\sqrt{\{a+bx-(cx)^2\}}} = \frac{1}{\sqrt{c}} \sin^{-1} \frac{2cx-b}{\sqrt{(4ac+b^2)}}$$

7. $$\int \sqrt{(a+bx-cx^2)}\, dx$$

$$= \sqrt{c}\left\{\frac{2cx-b}{4c}\sqrt{(a+bx-cx^2)} + \frac{4ac+b^2}{8c^2}\sin^{-1}\frac{2cx-b}{\sqrt{(4ac+b^2)}}\right\}$$

8. $$\int \frac{dx}{x\sqrt{(a^2 \pm x^2)}} = \frac{1}{a} \log \frac{x}{a+\sqrt{(a^2 \pm x^2)}}$$

9. $$\int \frac{dx}{a+bx+cx^2} = \frac{1}{c} \int \frac{dx}{\left(x+\frac{b}{2c}\right)^2+\frac{4ac-b^2}{4c^2}}$$

If $\frac{4ac-b^2}{4c^2}$ be negative, this becomes

$$\frac{1}{\sqrt{(b^2-4ac)}}\log\frac{2cx+b-\sqrt{(b^2-4ac)}}{2cx+b+\sqrt{(b^2-4ac)}}$$

If $\frac{4ac-b^2}{4c^2}$ be positive, then the integral is

$$\frac{2}{\sqrt{(4ac-b^2)}}\tan^{-1}\frac{2cx+b}{\sqrt{(4ac-b^2)}}$$

10. $\displaystyle\int_a^{2a}\frac{dx}{\sqrt{(x^2+a^2)}}=\log\frac{2+\sqrt{5}}{1+\sqrt{2}}$

11. $\displaystyle\int\sqrt{\left(\frac{m+x}{x}\right)}dx$

$$=\sqrt{(mx+x^2)}+m\log\{\sqrt{x}+\sqrt{(m+x)}\}$$

12. $\displaystyle\int\frac{dx}{x+\sqrt{(x^2-1)}}$

$$=\frac{x^2}{2}-\frac{x\sqrt{(x^2-1)}}{2}+\tfrac{1}{2}\log\{x+\sqrt{(x^2-1)}\}$$

13. $\displaystyle\int_0^{2a}\text{vers}^{-1}\frac{x}{a}\,dx=\pi a$

14. $\displaystyle\int_0^{2a}x^2\,\text{vers}^{-1}\frac{x}{a}\,dx=\frac{11\pi a^3}{16}$

15. $\displaystyle\int\frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}\,dx=\sin^{-1}x\frac{x}{\sqrt{(1-x^2)}}+\log\sqrt{(1-x^2)}$

16. $\displaystyle\int_0^a\frac{x^{\frac{5}{2}}\,dx}{\sqrt{(a-x)}}=\frac{5\pi a^3}{16}$

17. $\displaystyle\int\sqrt{(2ax-x^2)}\,dx=\frac{x-a}{2}\sqrt{(2ax-x^2)}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)$

18. $\displaystyle\int\sqrt{(2ax-x^2)}\,x\,dx$

$$=-\tfrac{1}{3}(2ax-x^2)^{\frac{3}{2}}+a\int\sqrt{(2ax-x^2)}\,dx$$

19. $\displaystyle\int\sqrt{(2ax+x^2)}\,dx$

$$=\frac{(x+a)\sqrt{(2ax+x^2)}}{2}-\frac{a^2}{2}\log\left\{x+a+\sqrt{(2ax+x^2)}\right\}$$

20. $\displaystyle\int_0^{2a}\sqrt{(2ax-x^2)}\,x^2\,dx=\frac{5\pi a^4}{8}$

21. $\displaystyle\int\frac{dx}{\sqrt{(2ax+x^2)}}=\log\{(x+a)+\sqrt{(2ax+x^2)}\}$

22. $\displaystyle\int\frac{dx}{\sqrt{(1-3x-x^2)}}=\sin^{-1}\frac{3+2x}{\sqrt{13}}$

23. $$\int \frac{dx}{\sqrt{(x^2 - 6x + 13)}} = \log\{x - 3 + \sqrt{(x^2 - 6x + 13)}\}$$

24. $$\int \frac{x^3\,dx}{\sqrt{(x-1)}} = 2\sqrt{(x-1)}\left\{\frac{(x-1)^3}{7} + \tfrac{3}{5}(x-1)^2 + x\right\}$$

25. $$\int_0^{2a} \text{vers}^{-1}\frac{x}{a}\,dx = \pi a$$

26. $$\int \frac{\sqrt{(1-x^2)}}{x^4}\sin^{-1}x\,dx = -\frac{\sin^{-1}x(1-x^2)^{\frac{3}{2}}}{3x^3} - \frac{1}{6x^2} - \frac{\log x}{3}$$

27. $$\int \frac{\sin^2 x\,dx}{a + b\cos^2 x} = \left(\frac{a+b}{ab^2}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{\sqrt{a}\tan x}{\sqrt{(a+b)}}\right) - \frac{x}{b}$$

28. $$\int x^3\sqrt{(a + bx^2)}\,dx = \left(\frac{a + bx^2}{5b^2} - \frac{a}{3b^2}\right)(a + bx^2)^{\frac{3}{2}}$$

29. $$\int_0^{\infty} \frac{dx}{(a^2 + x^2)(b^2 + x^2)} = \frac{\pi}{2ab(a+b)}$$

30. $$\int_0^{\frac{\pi}{2}} \frac{\sin^3 x\,dx}{1 + c\cos x} = \frac{c^2 - 1}{c^3}\log(1 + c) + \frac{2 - c}{2c^2}$$

## LENGTHS OF CURVES.

**20.** Let $s$ denote the length of any arc of a plane curve, and $x$ and $y$ the rectangular co-ordinates of any point of the curve; then

$$(ds)^2 = (dx)^2 + (dy)^2.$$

and

$$ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx = \left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}} dy$$

Therefore, if $x_1$, $y_1$, and $x_2$, $y_2$, are the co-ordinates of any two points of the curve, and if

$$s\Big]_{x_2}^{x_1} \text{ or } s\Big]_{y_2}^{y_1}$$

denote the length of the arc between the two points, then

$$s\Big]_{x_2}^{x_1} = \int_{x_2}^{x_1}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx = s\Big]_{y_2}^{y_1} = \int_{y_2}^{y_1}\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}} dy$$

### EXAMPLES.

1. To find the length of any arc of a parobola measuring from the vertex

$$y = \sqrt{(4ax)}$$

$$\frac{dy}{dx} = \sqrt{\frac{a}{x}}$$

$$s\Big]_0^x = \int_0 \left(\frac{x+a}{x}\right)^{\frac{1}{2}} dx = \sqrt{(ax + x^2)} + a \log \frac{\sqrt{x} + \sqrt{(x+a)}}{\sqrt{a}}$$

2. $y = a \cos^{-1}\left(\frac{a-x}{a}\right) + \sqrt{(2ax - x^2.)}$ (Cycloid.)

$$s\Big]_0^x = \sqrt{(8ax)}$$

## POLAR FORMULÆ.

**21.** Applying the general equations of transformation from rectangular to polar co-ordinates,

$$x = r \cos \theta$$

and

$$y = r \sin \theta$$

and, substituting in the formulæ of the previous article, we have

$$s\Big]_{\theta_2}^{\theta_1} = \int_{\theta_2}^{\theta_1} \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} d\theta = s\Big]_{r_2}^{r_1} = \int_{r_2}^{r_1} \left\{ 1 + \left(\frac{r\, d\theta}{dr}\right)^2 \right\}^{\frac{1}{2}} dr$$

### EXAMPLES.

1. Show that the circumference of a circle is $2\pi a$.

2. $r = a(1 + \cos\theta)$. (Cardioide.)

Perimeter $= 8a$

## AREAS OF PLANE CURVES.

**22.** Let $x$ and $y$ be the rectangular co-ordinates of any point of a plane curve, and $x_1, y_1$, and $x_2, y_2$, the co-ordinates of any two given points of the curve; then, if $A_x$ denote the area enclosed between the curve, the axis of X, and the two ordinates $y_1$ and $y_2$,

$$A_x = \int_{x_2}^{x_1} y\, dx$$

and if $A_y$ denote the area enclosed between the curve, the axis of Y, and the two lines drawn through the given points parallel to the axis of X, then

$$A_y = \int_{y_2}^{y_1} x\,dy$$

**23.** In applying these formulæ, we have only to substitute in them the value of $y$ or $x$ obtained from the equation to the curve; thus, the equation to the parabola is

$$y^2 = 4ax$$

whence

$$y = \pm 2\sqrt{(ax)}$$

$$x = \frac{y^2}{4a}$$

$$A_x = \pm 2\sqrt{a}\int_{x_2}^{x_1}\sqrt{x}\,dx = \pm\frac{4\sqrt{a}}{3}\left\{x_1^{\frac{3}{2}} - x_2^{\frac{3}{2}}\right\}$$

$$A_y = \frac{1}{4a}\int_{y_2}^{y_1} y^2\,dy = \frac{1}{12a}\left\{y_1^3 - y_2^3\right\}$$

It may sometimes be more convenient to substitute the values of $dx$ or $dy$ obtained from the equation to the curve. In this case, the *limits* of the integral must be changed from $x_1$, $x_2$, to the corresponding values of $y$, ($y_1$, $y_2$,) or *vice versa;* that is, the limits must always be values of the independent variable.

## SIGN OF THE AREA.

**24.** $\int_{x_2}^{x_1} y\,dx$ will be positive or negative according as $y$ and $dx$ have the same or contrary signs.

Now, $y\,dx$, the differential of $A_x$, was found upon the supposition that this area was generated by a variable ordinate, moving uniformly along the axis of X, toward the right; so that, in this case, $dx$ is always positive, and the sign of the area depends upon that of $y$. Therefore, if the area represented by $\int_{x_2}^{x_1} y\,dx$ lies *above* the axis of X, it is *positive;* if *below*, it is *negative.*

Similarly, if the area represented by $\int_{y_2}^{y_1} x\,dy$ lies to the *right* of the axis of Y, it is *positive;* and, if to the *left*, it is *negative.*

In finding the whole area of any curve, the sum of the several parts is taken, irrespective of sign, or as if they were all positive.

## LIMITS OF THE INTEGRATION.

**25.** From the preceding article, it follows that, when a curve crosses the axis of X, $\int_{x_2}^{x_1} y\,dx$ changes sign; and when it crosses the axis of Y, $\int_{y_2}^{y_1} x\,dy$ changes sign; therefore, if we include *between* the limits of the former any value of $x$ for which $y$ becomes 0, or *between* the limits of the latter any value of $y$ for which $x$ becomes 0, we shall, in general, obtain a difference of two areas instead of their sum. Thus, the equation to a curve is

$$y = x^3 + 4x^2$$

then

$$A_x\Big]_{x_2}^{x_1} = \int_{x_2}^{x_1} (x^3 + 4x^2)\,dx$$

$$= \frac{x^4}{4} + \frac{4x^3}{3}\Big]_{x_2}^{x_1}$$

whence

$$A_x\Big]_{0}^{4} = 149\tfrac{1}{3}$$

$$A_x\Big]_{-4}^{0} = 21\tfrac{1}{3}$$

$$A_x\Big]_{-8}^{-4} = -362\tfrac{2}{3}$$

and the whole area, between $x = 4$ and $x = -8$, is

$$149\tfrac{1}{3} + 21\tfrac{1}{3} + 362\tfrac{2}{3} = 533\tfrac{1}{3}$$

If we had integrated between $x = 4$ and $x = -8$ at one operation, we would have obtained

$$A_x\Big]_{-8}^{4} = -192$$

which is the *algebraic* sum of the areas. In this case, we might have integrated between $x = 4$ and $x = -4$; for, although the ordinate at the origin is 0, the curve does not cross the axis of X at that point, and therefore the area does not change sign. Care must also be taken not to assign limits between which the value of the variable becomes infinite. In the following examples, $A$ denotes the whole area of a closed curve. *The curves should always be traced.*

## EXAMPLES.

### VIII.

1. $y^2 = 4x + 8y - 16$

$$A_x\Big]_0^4 = 26\tfrac{2}{3} \text{ or } 5\tfrac{1}{3}$$

$$A_y\Big]_0^8 = 10\tfrac{2}{3}$$

2. $y^2 = 4x + 8y$

$$A_x\Big]_0^{12} = 122\tfrac{1}{3} \text{ or } -26\tfrac{1}{3}$$

$$A_y\Big]_0^8 = -21\tfrac{1}{3}$$

3. $y = x^3 + ax^2$

$$A_x\Big]_0^a = \frac{7a^4}{12}$$

$$A_x\Big]_{-a}^0 = \frac{a^4}{12}$$

4. $y^2 = 2ax - x^2$

$$A = \pi a^2$$

5. $y^2 = 2ax + x^2$

$$A_x\Big]_0^{2a} = \pm a^2\{3\sqrt{2} - \tfrac{1}{2}\log(3 + 2\sqrt{2})\}$$

6. $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$

$$A = \tfrac{3}{8}\pi ab$$

7. $y(x^2 + a^2) = c^2(a - x)$

$$A_x\Big]_0^a = c^2\left(\frac{\pi}{4} - \tfrac{1}{2}\log 2\right)$$

8. $a^4y^2 + b^2x^4 = a^2b^2x^2$

$$A = \tfrac{4}{3}ab$$

9. Find the area of a loop of the curve

$$a^2y^4 = x^4(a^2 - x^2) \qquad \frac{4a^2}{5}$$

10. Find the area of a loop of the curve

$$y^2(a^2 + x^2) = x^2(a^2 - x^2) \qquad \frac{a^2}{2}(\pi - 2)$$

11. Find the area of the loop of the curve

$$2y^2(a^2 + x^2) = (a^2 - x^2)^2$$

$$a^2\{3\sqrt{2}\log(1 + \sqrt{2}) - 2\}$$

12. $y = x(4 - x)^2$

$$A_x\Big]_0^4 = 21\tfrac{1}{3}$$

13. $y(a^2 + x^2) = a^3$

$$A_x\Big]_0^a = \frac{\pi a^2}{4}$$

$$A_x\Big]_0^\infty = \frac{\pi a^2}{2}$$

14. $y = \sin x$

$$A_x\Big]_0^\pi = 2$$

$$A_x\Big]_\pi^{2\pi} = -2$$

15. $xy = m^2$

$$A_x\Big]_b^a = m^2 \log\left(\frac{a}{b}\right)$$

16. $y^2 = \dfrac{a^2}{(ax - x^2)}$

$$A_x\Big]_0^a = \pm \pi a$$

17. $(y - 1)^2 = x^4 - x^2$

$$A_x\Big]_2^3 = 1 \pm \tfrac{1}{3}\{\sqrt{512} - \sqrt{27}\}$$

18. $(x + y)^2 = 2(x - y)$

$$A_x\Big]_0^2 = \tfrac{1}{3} \text{ or } -8\tfrac{1}{3}$$

19. $y^2 = e^x$

$$A_x\Big]_0^2 = \pm 2(e - 1)$$

$$A_x\Big]_{-\infty}^0 = \pm 2$$

20. $x^2y^2(4x^2 - 9) = 4$

$$A_x\Big]_{\frac{3}{2}}^\infty = \pm \frac{\pi}{3}$$

21. $y = x \tan a - \dfrac{x^2}{4h \cos^2 a}$

$$A_x\Big]_0^{2h \sin 2a} = \tfrac{4}{3} h \sin 2a \, h \sin^2 a$$

22. $y^2(a - x) = x^5$

$$A_x\Big]_0^a = \pm \frac{5\pi a^3}{16}$$

23. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$$A_x\Big]_a^{2a} = \pm\, ba\{\sqrt{3} - \tfrac{1}{2}\log(2+\sqrt{3})\}$$

24. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$A = \pi ab$$

25. $y^2 = x^2(9-x^2)(x^2-4)^2$

$$A = \frac{80}{3}\sqrt{5}+\frac{72}{5}$$

## POLAR FORMULÆ.

**26.** Let $r$ and $\theta$ be the polar co-ordinates of any point of a plane curve, and $r_1$, $\theta_1$, and $r_2$, $\theta_2$, the co-ordinates of any two given points of the curve; then, if $A$ denote the area included between the curve and the two radii vectores $r_1$ and $r_2$,

$$A = \tfrac{1}{2}\int_{\theta_2}^{\theta_1} r^2\,d\theta = \tfrac{1}{2}\int_{r_2}^{r_1} r^2\,d\theta$$

The same precautions (as to assigning limits) as were mentioned for curves referred to rectangular axes must be observed in the use of these formulæ. Thus: required the whole area of the lemniscata, the equation to which is

$$r^2 = a^2\cos 2\theta$$

$$A = \frac{a^2}{2}\int_{\theta_2}^{\theta_1}\cos 2\theta\,d\theta$$

To find the whole area of the right-hand loop, we must integrate between the values of

$$\theta = \frac{\pi}{4} \text{ and } \theta = -\frac{\pi}{4}$$

Thus,

$$A = \frac{a^2}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2\theta\,d\theta = \frac{a^2}{2}\cdot\frac{\sin 2\theta}{2}\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{a^2}{2}$$

And as the curve is symmetrical, its entire area is $a^2$.

### EXAMPLES.

1. $r = 2a\cos\theta$ $\qquad A = \pi a^2$
2. $r = 2a(1-\cos\theta)$ $\qquad A = 6\pi a^2$
3. $r = 2a\cos\theta\cos 2\theta$ $\qquad A = \frac{\pi a^2}{2}$

## AREAS OF SURFACES OF SOLIDS OF REVOLUTION.

**27.** If the curve revolve about the axis of X,

$$S\Big]_{x_2}^{x_1}=2\pi\int_{x_2}^{x_1} y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx$$

if about the axis of Y,

$$S\Big]_{y_2}^{y_1}=2\pi\int_{y_2}^{y_1} x\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}}dy$$

### EXAMPLES.

1. Find the surface of a cylinder.
2. Find the surface of a cone.
3. Find the surface of a sphere.

## VOLUMES OF SOLIDS OF REVOLUTION.

**28.** If $V_x$ denote the volume of the solid generated by the revolution of the area $A_x$ about the axis of X,

$$V_x=\pi\int_{x_2}^{x_1} y^2\,dx$$

and, if $V_y$ denote the volume generated by the revolution of $A_y$ about the axis of Y,

$$V_y=\pi\int_{y_2}^{y_1} x^2\,dy$$

$V_x$ and $V_y$ are always positive; so that, in assigning limits to the integration, it is only necessary to see that they do not include imaginary values of the variables.

If we denote the volume of the solid formed by the revolution of $A_x$ about the axis of Y by $V_y A_x$, and the volume of the solid formed by the revolution of $A_y$ about X by $V_x A_y$, then

$$V_y A_x=2\pi\int_{x_2}^{x_1} xy\,dx$$

$$V_x A_y=2\pi\int_{y_2}^{y_1} xy\,dy$$

**29.** The equation to the parabola is $y^2=4ax$; whence

$$V_x\Big]_0^x=4\pi a\int_0 x\,dx=2\pi a x^2=\frac{\pi}{2}\,x\,y^2$$

and

$$V_y\Big]_0^y = \frac{\pi}{16a^2}\int_0^y y^4\,dy = \frac{\pi y^5}{80a^2} = \frac{\pi}{5}\,x^2\,y$$

$$V_y A_x\Big]_0^x = \pm\;4\pi\sqrt{a}\int_0^x x^{\frac{3}{2}}\,dx = \pm\;\frac{8\pi\sqrt{a}\,x^{\frac{5}{2}}}{5} = \pm\;\frac{4\pi}{5}\,x^2\,y$$

$$V_x A_y\Big]_0^y = \frac{\pi}{2a}\int_0^y y^3\,dy = \frac{\pi y^4}{8a} = \frac{\pi}{2}\,x\,y^2$$

In this case, $V_y\,A_x$ is affected by the double sign: the positive sign applies to the volume formed by the revolution of that part of the area which lies above the axis of X, and the negative sign to that formed by the revolution of the area below the axis of X. In general, $\int xy\,dx$ has the sign of $y$, and $\int yx\,dy$ has the sign of $x$. It will be seen that $V_y\,A_x\Big]_0^x$ might be found by subtracting $V_y\Big]_0^y$ from the cylinder formed by the revolution of the rectangle $xy$ about the axis of X.

### EXAMPLES.

### IX.

1. $y^2 = 2ax - x^2$

$$V_x = \tfrac{4}{3}\,\pi a^3$$

2. $a^2y^2 + b^2x^2 = a^2b^2$

$$V_x = \tfrac{4}{3}\,\pi b^2 a$$

$$V_y = \tfrac{4}{3}\,\pi a^2 b$$

3. $y^2\,(2a - x) = x^3$

$$V_x\Big]_0^a = 8a^3\pi\,[\log 2 - \tfrac{2}{3}]$$

4. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$$V_x\Big]_a^{2a} = \frac{4\pi}{3}\,ab^2$$

5. Show that the volume of a frustum of a sphere is

$$\frac{\pi h}{6}\,\{h^2 + 3\,(r_1^2 + r_2^2)\}$$

in which $h$ is the height and $r_1$ and $r_2$ are the radii of the ends.

6. The curve $y^2\,(2a - x) = x^3$ revolves round its asymptote; show that the volume generated is $2\pi^2a^3$.

7. The curve $xy^2 = 4a^2(2a - x)$ revolves about its asymptote; show that the volume generated is $4\pi^2a^3$.

8. Find the volume of a frustum of a cone, having given its altitude and the radii of the ends.

$$V = \frac{\pi h}{3}(r^2 + Rr + R^2)$$

9. $y^2 = x^2(9 - x^2)(x^2 - 4)^2$

$$V_x = 286\tfrac{34}{35}\pi$$

10. Find the volume of an anchor-ring; the internal radius being $(b - a)$, and the external $(b + a)$. $2\pi^2a^2b$

11. $x^2y^2 = a^2(ax - x^2)$

$$V_x\Big]_{\frac{a}{2}}^{a} = a^3\left\{\log 2 - \tfrac{1}{2}\right\}$$

## CENTRES OF GRAVITY.

**30.** Denoting by $\bar{x}$ the distance from the axis of Y of the centre of gravity of the area included between any plane curve and the axis of X, and by $\bar{y}$ the distance of the centre of gravity of the *same* area from the axis of X; then

$$\bar{x} = \frac{\int_{x_2}^{x_1} yx\,dx}{\int_{x_2}^{x_1} y\,dx}$$

and

$$\bar{y} = \frac{1}{2}\cdot\frac{\int_{x_2}^{x_1} y^2\,dx}{\int_{x_2}^{x_1} y\,dx}$$

where each integral must be obtained independently.

**31.** For instance, it is required to determine the position of the centre of gravity of a semi-parabola of abscissa $x$ and ordinate $y$,

$$y^2 = 4ax$$

$$\bar{x} = \frac{\pm 2\sqrt{a}\int_0 x^{\frac{3}{2}}\,dx}{\pm 2\sqrt{a}\int_0 x^{\frac{1}{2}}\,dx} = \frac{\frac{2}{5}x^{\frac{5}{2}}}{\frac{2}{3}x^{\frac{3}{2}}} = \tfrac{3}{5}x$$

$$\bar{y} = \frac{\sqrt{a}\int_0 x\,dx}{\pm\int_0 \sqrt{x}\,dx} = \pm\sqrt{a}\,\frac{\frac{x^2}{2}}{\frac{2}{3}x^{\frac{3}{2}}} = \pm\tfrac{3}{4}\sqrt{(ax)} = \pm\tfrac{3}{8}y$$

Here $\bar{x}$ is always positive, and $\bar{y}$ is positive or negative according as the centre of gravity of the area above or below the axis of X is required.

**32.** Denoting by $\bar{x}$ the distance from the axis of Y of the centre of gravity of a solid formed by the revolution of the area included between any plane curve and the axis of X about the axis of X; then

$$\bar{x} = \frac{\int_{x_2}^{x_1} y^2 x \, dx}{\int_{x_2}^{x_1} y^2 \, dx} \cdot$$

Thus the position of the centre of gravity of a paraboloid may be determined as follows:

$$y^2 = 4ax$$

$$\bar{x} = \frac{4a \int_0^x x^2 \, dx}{4a \int_0^x x \, dx} = \frac{\frac{x^3}{3}}{\frac{x^2}{2}} = \tfrac{2}{3} x \, .$$

And, of course, $\bar{x}$ fixes the position of the required centre of gravity, as the centre of gravity must lie in the axis of X. In the same way an integral expression may be determined for the distance from the axis of X of the centre of gravity of a solid formed by the revolution of an area about the axis of Y.

### EXAMPLES.

### X.

1. Centre of gravity of a triangle. $\tfrac{2}{3} h$ from apex.
2. Centre of gravity of a cone. $\tfrac{3}{4} h$ from apex.
3. Centre of gravity of a frustum of a cone.

$$\frac{h}{4} \cdot \frac{3R^2 + 2Rr + r^2}{R^2 + Rr + r^2}$$

4. Centre of gravity of a hemisphere. $\tfrac{3}{8} r$ from base.
5. Show that the volume of any solid of revolution is equal to the area of the generating figure multiplied by the length of the path described by the centre of gravity of this figure.
6. Centre of gravity of a semi-ellipsoid of revolution,

prolate, $\tfrac{3}{8} a$ from centre.
oblate, $\tfrac{3}{8} b$ from centre.

7. Find the distance below the surface of the water of the

centre of gravity of a quadrantal log-ship of uniform thickness and density, with its apex just in the surface and its medial line vertical. .6002$r$

8. Find a general integral expression for the distance of the centre of gravity (from the axis of Y) of the *surface* of a solid generated by the revolution of an area about the axis of X.

## SIMPSON'S RULE.

**33.** The primary object of this rule is to find an approximate expression for the area between a curve the axis of X and two given ordinates (the equation to the curve not being known), when we can, by measurement, determine the lengths of the two given ordinates, and of an equidistant intermediate ordinate, and also the length of the *common interval* between these three ordinates. Let $y_1$ denote the right-hand ordinate, $y_0$ the middle ordinate, and $y'$ the left-hand ordinate; and let $h$ denote the common interval. Let the unknown equation of the curve be

$$y = f(x)$$

which may be expanded by McLaurin's theorem, and written

$$y = A + Bx + Cx^2 + Dx^3 + R \tag{1}$$

where $R$ denotes the sum of the remaining terms containing powers of $x$ higher than the *third;* then denoting by $A_x$ the desired area,

$$A_x = \int_{-h}^{h} y\,dx = \int_{-h}^{h} (A + Bx + Cx^2 + Dx^3 + R)\,dx \tag{2}$$

$$= 2Ah + \frac{2Ch^3}{3} + \int_{-h}^{h} R\,dx$$

Now, $R$ is a function of $x$ containing only powers higher than the *third;* and, when $x$ is small, $R$ is so much smaller that it may be neglected.

$\int_{-h}^{h} R\,dx$ will consist of a series of terms of the *odd* powers of $h$, beginning with the *fifth;* and, if $h$ be taken small, $\int_{-h}^{h} R\,dx$ becomes so small that it may be neglected; therefore, assuming that $h$ is taken small,

$$A_x = 2Ah + \frac{2Ch^3}{3} \text{ (approximately)}$$

$$= h\left\{2A + \frac{2Ch^2}{3}\right\} \tag{3}$$

in which we have to express $A$ and $C$ in terms of known quantities. To do this, put $x = h$ in (1), remembering that, as $h$ is assumed to be small, $R$ may be neglected; then

$$y_1 = A + Bh + Ch^2 + Dh^3$$

putting $x = -h$,

$$y' = A - Bh + Ch^2 - Dh^3$$

then, by addition,

$$y' + y' = 2A + 2Ch^2$$

also, putting $x = 0$, $y_0 = A$. Hence

$$2Ch^2 = y_1 + y' - 2y_0$$

and, substituting in (3),

$$(4) \qquad A_x = \frac{h}{3}\{4y_0 + y_1 + y'\}$$

which is Simpson's rule.

It is evident that, if we take two adjoining portions of a curve included between equidistant ordinates, and determine their areas by this rule, then, by addition, the whole area

$$= \frac{h}{3}\{y_1 + 4y_2 + 2y_3 + 4y_4 + y_5\}$$

the ordinates being marked in order from either end; therefore, this rule may be extended to any *odd* number of ordinates; and, consequently, in determining any required area, by taking a sufficient number of ordinates, we can make $h$ as small as we please, and our expression for the area as accurate as we please.

The use of this rule may be extended to finding the volumes of solids by taking the areas of an odd number of equidistant cross-sections, and treating these areas exactly as the ordinates are treated in computing an area. The rule may also be applied to finding centres of gravity, &c., and, generally, to finding the approximate value of any integral.

## THE LOXODROMIC CURVE.

**34.** *A ship sails from the equator to a given latitude, $L$, keeping always on the same course; to find the difference of longitude between the point arrived at and the point left.*

Suppose the ship's track stereographically projected on the plane of the equator; then, by the principles of stereographic projection, the projection of the ship's track makes with the projection of each meridian an angle equal to that which the ship's track makes with the meridian itself; and, also, the distance of the projection of the ship at any instant from the centre of the primitive circle is equal to $R \tan \frac{1}{2}(90 - L)$, where $L$ is the latitude of the ship at that instant, and $R$ the radius of the sphere.

Let $\theta$ = angular difference of longitude passed through by ship;

$r$ = distance of projection of ship from the centre of the primitive circle; and

$C$ = course.

Then

$$\cot C = -\frac{dr}{r\,d\theta} \quad (dr \text{ being negative})$$

Integrating

$$\cot C \int_0^\theta d\theta = -\int_R^r \frac{dr}{r}$$

$$\cot C.\theta = \log \frac{R}{r} = \log \cot \tfrac{1}{2}(90 - L) = \log \tan \tfrac{1}{2}(90 + L)$$

$$\text{D} = \text{dif. long.} = R\theta = R \tan C \log \tan (\tfrac{1}{2} L + 45^\circ)$$

but in this the logarithm is Napierian; therefore, $R$ denoting the radius of the sphere in nautical miles,

$$\text{D} = R \tan C \log_e 10 \log_{10} \tan (\tfrac{1}{2} L + 45^\circ)$$

## CENTRES OF PRESSURE.

**35.** The general expression for the distance, below the surface of a liquid, of the centre of pressure on any immersed surface, is

$$\bar{p} = \frac{\int_{y_2}^{y_1} y^2 x\,dy}{\int_{y_2}^{y_1} y\,x\,dy}$$

where the axis of Y is vertical, and the axis of X is in the surface of the fluid.

### EXAMPLES.

### XI.

1. The centre of pressure on a rectangle, with one edge in the surface.

2. The centre of pressure on a trapezoid, with one of the parallel sides in the surface.

3. The centre of pressure of a semicircle, with its base in the surface.

4. A rectangular flood-gate is 5′ in height, and opens by turning on a horizontal axis 2′ 4″ from the bottom of the gate; how high will the water rise above the top of the gate before its pressure opens it? Ans. 10′

5. A scuttle-butt has for its base an ellipse whose diameters are 52″ and 28″, and for its top an ellipse whose diameters are 40″ and 22″, and its height is 24″; required the number of wine-gallons of 231 cubic inches it will contain.

(To be solved by integration.) 94.33 gallons.

6. Solve example 5 by Simpson's rule, and explain why Simpson's rule gives an exactly correct result in this case.

7. Find the volume of the solid cut off from a right circular cylinder by a plane passing through the centre of the base, and inclined at an angle $a$ to the plane of the base.

$$\frac{2}{3} r^3 \tan a$$

○

## *CORRECTIONS.*

*Page* 12. *Ex* 82. $(x^6—a^2)$ should be $(x^6—a^2)^{\frac{1}{2}}$.

14. 9. Insert a bracket before $ax$.

14. 20. $3x^2$ should be $3x^3$.

15. *Equation next before examples.* Insert $dx$ after sin $x$.

16. *Ex.* 9. Insert $dx$.

20. 10. 6 log $(x+1)$ should be 6 log $(x—1)$.

20. 16. Insert $dx$.

29. 5 and 7. The original radical, wherever it occurs in the result, should be divided by $c^{\frac{1}{2}}$.

29. 6. $(cx)^2$ should be $cx^2$.

30 14. The 16 in the result should be 6.